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SHORTER COMMUNICATIONS

LINEARIZED BUOYANT MOTION DUE TO IMPULSIVELY HEATED VERTICAL PLATE(S)

R. T. DOTY and M. C. JISCHKE
McNeese State University, Lake Charles, Louisiana, U.S.A.
and
University of Oklahoma, Norman, Oklahoma, U.S.A.

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NOMENCLATURE

g, acceleration of gravity;
$H(t)$, Heaviside function;
k, thermal conductivity;
L, width of slot;
p, pressure;
Pr, Prandtl number;
Ra, Rayleigh number;
$S(x, t)$, reduced stream function variable;
T, temperature;
t, time;
t_h, heat-up time;
u, w, horizontal and vertical velocity components;
V, reduced temperature variable;
x, z, horizontal and vertical coordinates;
β, thermal expansion coefficient;
ΔT, temperature difference in basic state;
ε, small parameter;
ρ, density;
σ, stretched stream function variable, $Ra^{1/4}S$;
τ, stretched time variable, $Ra^{-1/4}t$;
ξ, boundary-layer coordinate $Ra^{1/4}(\frac{1}{2} \pm x)/\sqrt{2}$;
ψ, stream function.

Subscript

0, basic state, lowest order.

Superscript

B, boundary layer;
I, inviscid interior.

THE FORMATION of natural convection boundary layers is revealed by an exact solution the the linearized equations governing the motion of a fluid bounded by an impulsively heated infinite vertical plate. This result provides the motivation for a second calculation which describes the convection process by which a stratified fluid in a vertical slot is heated as a result of an impulsive change in the stratification of the two boundary plates. The result is an asymptotic solution for large Rayleigh and arbitrary Prandtl number.

SINGLE VERTICAL PLATE

Consider first a compressible Newtonian fluid confined to $x > 0$ by an infinite plate at $x = 0$. Given initially that the plate has a temperature that varies linearly along the plate and that the fluid is in static equilibrium with the plate, the problem is to find the flow field that results from impulsively changing the plate stratification by a small amount.

Let z be the coordinate measured along the plate such that gravity, g, acts in the negative z-direction and $z = 0$ corresponds to the point about which the initial plate temperature profile is impulsively rotated. Further, let L be a reference length such that the plate temperature is given by $[1+\varepsilon H(t)]\Delta Tz/L$ where $\varepsilon \ll 1$, $H(t)$ is the Heaviside function, and $\Delta T/L$ is the stratification of the basic state.

Introduce the following normalized variables (denoted by an asterisk) appropriate to buoyant, weakly-stratified, small-disturbance flow:

$$\nabla^* \equiv L\nabla,\ t^* = \left(\frac{g\beta_0 \Delta T Pr}{L}\right)^{1/2} t,\ \mathrm{V}^* = \frac{\mathrm{V}'}{\varepsilon}\left(\frac{Pr}{gL\beta_0\Delta T}\right)^{1/2} \tag{1}$$

$$T^* = \frac{T'}{\varepsilon\Delta T},\quad \rho^* = \frac{\rho'}{\varepsilon\rho_0\beta_0\Delta T},\quad p^* = \frac{p'}{\varepsilon\rho_0 gL\beta_0\Delta T}.$$

Pressure, density, temperature, velocity, time, and thermal expansion coefficient are given by p, ρ, T, V, $\mathbf{t}$ and β respectively. The subscript zero denotes references values in the basic state of static equilibrium and the prime indicates a small perturbation. The Prandtl number $Pr = \mu Cp_0/k$ in which the viscosity, μ, and thermal conductivity, k, are both assumed constant, and Cp is the specific heat at constant pressure.

With these definitions the governing equations are (dropping the asterisk notation)

Mass
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2}$$

x-Momentum
$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + Ra^{-1/2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{3}$$

z-Momentum
$$\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + T + Ra^{-1/2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{4}$$

Energy
$$Pr\frac{\partial T}{\partial t} + w = Ra^{-1/2}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{5}$$

neglecting dissipation. Here u and w represent velocities in the x and z directions respectively. The Rayleigh number $Ra = PrgL^3\rho_0\beta_0\Delta T/\mu^2$. These equations represent the so-called "bousinesq" approximation for temperature driven, weakly stratified, small-disturbance flow. The Bousinesq approximation assumes that changes in density are important only in the buoyant force as it appears in the momentum equation.

The associated initial and boundary conditions are

$$t = 0:\quad T = u = w = 0 \tag{6}$$

$$x = 0:\quad T = H(t)z,\quad u = w = 0 \tag{7}$$

$$x \to \infty:\quad T, w \to 0. \tag{8}$$

There is no vertical length scale inherent in the geometry, thus the linear vertical spatial dependence that the temperature exhibits at the boundary can be assumed to hold throughout the fluid, $T = zV(x, t)$. Introduce the stream-function ψ as

$$\psi = -zS(x, t). \tag{9}$$

The functions S and V then satisfy the following coupled pair of linear partial differential equations

$$Ra^{1/2}\left(\frac{\partial}{\partial t} - Ra^{-1/2}\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 S}{\partial x^2} = \frac{\partial V}{\partial x} \tag{10}$$

$$\left(Pr\frac{\partial}{\partial t} - Ra^{-1/2}\frac{\partial^2}{\partial x^2}\right)V = -Ra^{1/4}\frac{\partial S}{\partial x}. \tag{11}$$

These equations are ideally suited for solution by means of Laplace transforms. When the Prandtl number is unity, the resulting transformed expression can be inverted analytically [1]. There results

$$V = \mathrm{Real}[\exp(-i^{1/2}\xi) + F_1(\xi, t)] \tag{12}$$

$$Ra^{1/4}S = \tfrac{1}{2}\mathrm{Imag}[\{\exp(-i^{1/2}\xi) - \mathrm{erf}((it)^{1/2}) - F_2(\xi, t)\}/i^{1/2}] \tag{13}$$

where $\xi = Ra^{1/4}x$ and

$$F_{1,2}(\xi, t) = \tfrac{1}{2}[\exp(i^{1/2}\xi)\,\mathrm{erfc}((it)^{1/2} + \xi/(2t^{1/2})) \mp \exp(-i^{1/2}\xi)\,\mathrm{erfc}((it)^{1/2} - \xi/(2t^{1/2}))]. \tag{14}$$

When Prandtl number is not unity, the transformed expressions must be inverted numerically. The contour integrals in the complex plane can be reduced to real quadratures which are evaluated numerically to give "exact" solutions for the impulsively heated infinite vertical plate.

Of particular interest is the final steady state where the temperature, velocity, and pressure are given by

$$T/z = \exp(-\bar{\xi})\cos\bar{\xi} \tag{15}$$

$$w/z = \exp(-\bar{\xi})\sin\bar{\xi} \tag{16}$$

$$-(\sqrt{2})Ra^{1/4}u = 1 - \exp(-\bar{\xi})(\sin\bar{\xi} + \cos\bar{\xi}) \tag{17}$$

$$Ra^{1/2}p = \exp(-\bar{\xi})\sin\bar{\xi} \tag{18}$$

where $\bar{\xi} = \xi/2^{1/2}$. These steady state solutions are shown in Fig. 1 and reveal that the plate temperature change is communicated to the fluid through viscous and thermal boundary layers whose thicknesses are of order $Ra^{-1/4}$. Within the thin viscous layer the buoyant force is balanced by viscous shear stresses and the pressure is constant to order $Ra^{-1/4}$. Thus, in steady flow, fluid particles near the upper half of the plate are heated as a direct result of conduction and the increased buoyant force causes the fluid to rise near the plate with a vertical velocity component of order unity forming a boundary layer. Near the lower half of the plate the fluid particles are cooled, the buoyant force decreases, and the fluid settles near the plate. To compensate for the vertical mass flow in the boundary layer, a small horizontal mass flux of order $Ra^{-1/4}$ is induced in the inviscid core. This is the familiar plugging phenomena of stratified flow and here results directly from a boundary-layer-induced suction.

Examination of the transient solution shows that these boundary layers form on a time scale given by the inverse of the Brunt–Vaisala frequency $(g\beta_0\,\mathrm{d}T_0/\mathrm{d}z)^{1/2}$ corresponding to the time it would take a simple pendulum of unit length to make a few oscillations in a "stratified-reduced" gravity field $g\beta_0\,\mathrm{d}T_0/\mathrm{d}z$. The boundary-layer formation is complicated by viscous diffusion and minor oscillations due to excitation of the natural inviscid frequency. Figure 2 shows the development of the steady state temperature profile for Prandtl number unity and how the temperature profile tends to its steady state form in an oscillatory manner.

The effect of Prandtl number on the transient solution for temperature is shown in Fig. 3 where the temperature profile at a dimensional time equal to twice the inverse Brunt–Vaisala frequency is given. This result as well as the details of the analysis shows that the boundary-layer formation time increases as the square root of Prandtl number for large Prandtl number.

TWO VERTICAL PLATES—THE HEAT-UP PROBLEM

Consider next the effect of a second plate at $x = L$ so that the fluid is confined to $0 < x < L$. Again, the plates initially have a temperature that varies linearly along the plates and the fluid is in static equilibrium with the plates. At time $t = 0$, the stratification of the two plates is impulsively increased by a small amount. The problem is to find the flowfield that results and to determine the time required for the enclosed fluid to attain the stable state of increased stratification—the so-called heat-up time. The governing equations and boundary conditions remain the same as in the earlier case of a single impulsively heated plate except that the boundary condition at infinity is replaced by the following condition at $x = 1$.

$$x = 1: \quad T = H(t)z, \quad u = w = 0. \tag{19}$$

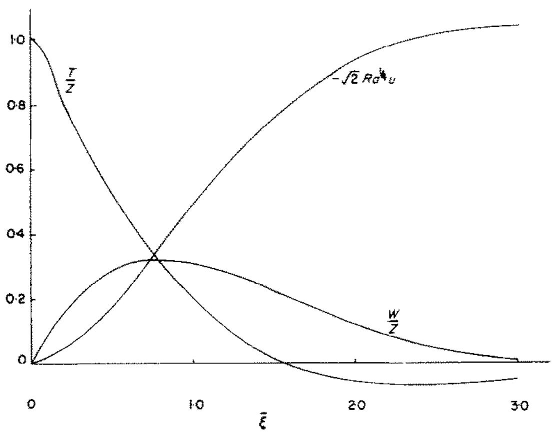

FIG. 1. Steady state velocity and temperature profiles.

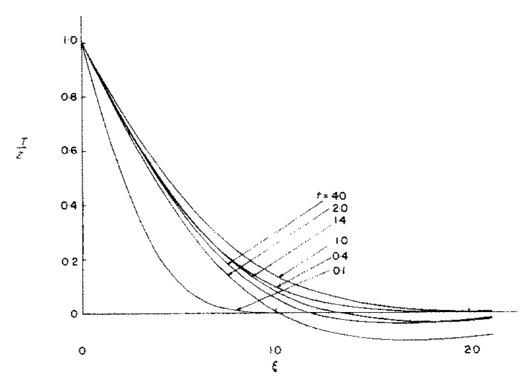

FIG. 2. Time evolution of temperature profile ($Pr = 1$).

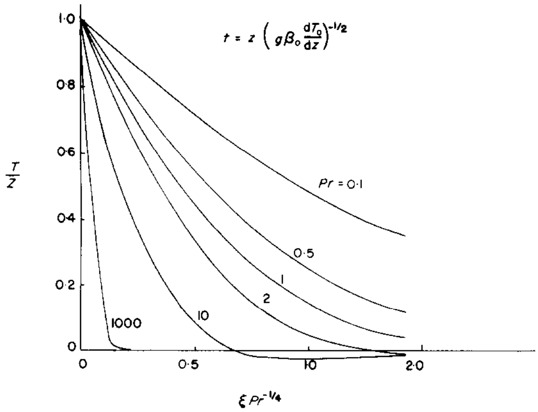

FIG. 3. Effect of Prandtl number on transient temperature profile.

For many problems of interest, the Rayleigh number is quite large suggesting the use of the methods of singular perturbation theory. This proves to be convenient as an exact solution of equations (10) and (11) with boundary conditions given by equations (6), (7) and (19) by, say, Laplace transforms is quite cumbersome and the asymptotic solutions for large Rayleigh number do describe the flow accurately and simply.

To motivate the form of the asymptotic solution, the nature of the heat-up process shall be briefly described. As our earlier analysis shows, the initial impulsive change in the boundary stratification produces a thermal layer at each plate which then starts to thicken by means of thermal diffusion. The increased buoyancy near the wall generates a velocity boundary layer and, within a time t of order unity corresponding to a dimensional time of the order of the inverse Brunt–Vaisala frequency, a quasi-steady boundary layer with thickness of the order of $Ra^{-1/4}$ develops. To compensate for the vertical mass flow in the boundary layer, a small horizontal mass flux of order $Ra^{-1/4}$ forms in the inviscid interior region. Mass conservation then implies that this small horizontal inflow into the boundary layers can be maintained only through the establishment of an equally small vertical interior motion. Since the interior flow is practically inviscid, the internal energy (here proportional to the temperature) of a fluid element which moves downward to replace the fluid entering the boundary layer is conserved. Thus fluid elements in the interior are convected to regions where the temperature was initially less than that of the fluid element. This is the convection process by which the inviscid interior is heated. As the inviscid interior approaches the new state of static equilibrium, the boundary layers decay. This happens in a dimensionless time of order $Ra^{1/4}$, a time scale not long enough for the boundary layers to thicken appreciably. The small oscillations set up by the initial impulse are inconsequential to the heat-up process and decay by viscous and thermal diffusion in a dimensionless time of order $Ra^{1/2}$.

Thus $Ra^{1/4}$ emerges as the time scale of the stream function. We therefore introduce "stretched" time $\tau = Ra^{-1/4}t$ and stream function $\sigma = Ra^{1/4}S$ variables and seek an asymptotic solution for Rayleigh number large and τ, V, and σ of order unity. The equations for V and σ then become

$$Ra^{-1/2}\left(\frac{\partial}{\partial\tau} - Ra^{-1/4}\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2\sigma}{\partial x^2} = V_x \quad (20)$$

$$\left(Pr\frac{\partial}{\partial\tau} - Ra^{-1/4}\frac{\partial^2}{\partial x^2}\right)V = -\sigma_x \quad (21)$$

with boundary conditions

$$x = \pm\tfrac{1}{2}: \quad V = 1, \quad \sigma, \frac{\partial\sigma}{\partial x} = 0. \quad (22)$$

We now decompose the stream function and temperature into interior and boundary-layer conditions

$$\begin{aligned}\sigma &= \sigma^I + \sigma^B \\ V &= V^I + V^B\end{aligned} \quad (23)$$

where σ^B and V^B are exponentially small away from the boundary.

For x of order unity, the interior solutions for Rayleigh number large are, to lowest order (denoted here by the subscript zero),

$$V_0^I = V_0^I(\tau) \quad (24)$$

$$\sigma_0^I = -xPr\frac{\mathrm{d}V_0^I(\tau)}{\mathrm{d}\tau} + f_0^I(\tau) \quad (25)$$

in terms of, as yet unknown, functions V_0^I and f_0^I.

To investigate the boundary layers near the plates we introduce a stretched coordinate $\xi = Ra^{1/4}(\frac{1}{2}\pm x)/\sqrt{2}$ for the left and right boundaries, respectively. The lowest order solution for Rayleigh number large and ξ of order unity are

$$V_0^B = (1 - V_0^I(\tau))\exp(-\xi)\cos(\xi) \quad (26)$$

$$\sigma_0^B = \pm\frac{(1 - V_0^I(\tau))}{\sqrt{2}}\exp(-\xi)(\cos\xi + \sin\xi) \quad (27)$$

for the left and right boundaries, respectively.

The no-slip boundary condition at the plates ($\xi = 0$) means that σ must vanish there. To lowest order, then, $(\sigma_0^B + \sigma_0^I)$ must vanish at $\xi = 0$. Thus, on the left plate,

$$\frac{Pr}{2}\frac{\mathrm{d}V_0^I(\tau)}{\mathrm{d}\tau} + f_0^I(\tau) - \frac{(1 - V_0^I(\tau))}{\sqrt{2}} = 0 \quad (28)$$

while on the right plate

$$-\frac{Pr}{2}\frac{\mathrm{d}V_0^I(\tau)}{\mathrm{d}\tau} + f_0^I(\tau) + \frac{(1 - V_0^I(\tau))}{\sqrt{2}} = 0. \quad (29)$$

Hence f_0^I must vanish identically and

$$V_0^I = 1 - \exp(-(\sqrt{2})\tau/Pr) \quad (30)$$

where we have made use of the fact that $V_0^I(0)$ must vanish. Equation (25) then allows determination of σ_0^I as

$$\sigma_0^I = -(\sqrt{2})\exp(-(\sqrt{2})\tau/Pr). \quad (31)$$

Thus we see that the interior is heated uniformly in x by a convection process which arises because of boundary-layer suction.

It should be noted that although this boundary-layer analysis allows V^I to satisfy the correct initial condition ($V^I = 0$), this is not the case for the stream function σ^I. Here the initial value of σ^I corresponds to the state of motion just after the establishment of the plate boundary layers. This is not surprising in that the asymptotic analysis is valid only for τ of order unity corresponding to

$$1 < t < Ra^{1/4} \quad (32)$$

i.e. just after the plate boundary layers form.

From these results we can conclude that the dimensional heat-up time (more accurately, the *e*-folding time) is given by

$$t_h = Pr^{1/2}Ra^{1/4}\left(\frac{2L}{g\beta_0\Delta T}\right)^{1/2} \tag{33}$$

which is much less than a diffusion time (which scales as $Ra^{1/2}$). From equation (33), one can show that the heat-up time for an insulating air gap in a pane of thermal glass is of the order of seconds, for the liquid oxygen in a spacecraft fuel tank it is of the order of a few hours, for the liquified natural gas in typical land-based storage tanks it is of a few days, while for the earth's mantle, the heat-up time may be of the order of 10^9 yr. This last figure is very approximate as it depends upon the properties of the earth's mantle which are not accurately known. It does suggest, however, that the convection patterns in the earth's mantle, which are presumably responsible for continental drift, may not have yet reached steady state.

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LAMINAR AND TURBULENT HEAT TRANSFER BY NATURAL CONVECTION

G. D. RAITHBY and K. G. T. HOLLANDS

Thermal Engineering Group, Department of Mechanical Engineering, University of Waterloo, Waterloo, Ontario, Canada

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NOMENCLATURE

- b, gap width between concentric cylinders or spheres;
- C_l, C_t, constants, see (2) and (5);
- D, diameter of cylinder or sphere;
- D_i, D_o, inside and outside diameters respectively of concentric cylinder or sphere;
- g, gravitational acceleration;
- g_s, component of gravitational acceleration along the surface;
- k, thermal conductivity;
- k_{eff}, effective thermal conductivity of fluid in gap between concentric cylinders or spheres;
- L, length of plate in flow direction;
- $\overline{Nu}_s$, average Nusselt number based on the relevant dimension, s;
- Pr, Prandtl number;
- s, curvilinear distance along surface from stagnation point;
- T_s, surface temperature;
- T_∞, fluid temperature far from surface;
- r, horizontal distance from axis of symmetry to point on surface;
- Ra_s, Ra_r, Rayleigh number based on $(T_s - T_\infty)$ and dimension s and r, respectively;
- β, thermal expansion coefficient;
- Δ_l, Δ_t, laminar and turbulent conduction thickness;
- κ, thermal diffusivity of fluid;
- ν, kinematic viscosity of fluid.

INTRODUCTION

THERE is a striking similarity between the processes governing the growth of a liquid condensate film, and the growth of the inner region of a natural convection boundary layer. Exploitation of this analogy has enabled the authors to obtain a general approximate solution to a broad class of free convection problems, predicting heat-transfer rates in good agreement with experimental results. The detailed development of this method and the comparison of predictions and measurements for several problems will appear elsewhere [1]. This note is intended to draw the reader's attention to this method, and to summarize the results.

In the main, the method applies to the problem of assessing the heat transfer from the external surfaces of single two-dimensional or axisymmetric bodies immersed in an extensive fluid, although enclosure problems are also considered in [1]. The method is first developed and tested for the case where the flow is laminar over the entire surface of the body. It is then extended to the turbulent case. A simple criterion is then proposed to predict the extent of the surface subjected respectively to laminar and turbulent heat transfer.

LAMINAR HEAT TRANSFER

The velocity extremum in a free convection laminar boundary layer divides the flow into two regions: the inner region adjacent to the wall, and the outer region. A central premise of the present model is that, in the inner region, inertial forces are not important and energy transfer normal to the walls is by conduction only. It is also hypothesized that the fraction of the boundary layer's total buoyancy carried by the inner region is invariant with s. With these assumptions, the rate of growth of the thickness of the inner region is found to be completely fixed locally, in a manner similar to the growth of a condensate film. As a consequence, the heat transfer can be calculated directly by integrating along the surface. The resultant expression for the heat transfer is derived in [1]. Expressed in terms of the local conduction thickness (defined as that thickness of stagnant fluid offering the same resistance to heat transfer as that